

Constraining Post-Newtonian $f(R)$ Gravity in the Solar System

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Received _____; accepted _____

ABSTRACT

We consider some models of $f(R)$ gravity that can be used to describe, in a suitable weak-field limit, the gravitational field of the Sun. Using a perturbative approach, we focus on the impact that the modifications of the gravitational field, due to the non-linearity of the gravity Lagrangian, have on the Solar System dynamics. We compare the theoretical predictions for the precession of the longitude of the pericentre ϖ of a test particle with the corrections to the standard Newtonian-Einsteinian precessions of the longitudes of perihelia of some planets of the Solar System recently estimated by E.V. Pitjeva by fitting large data sets with various versions of the EPM ephemerides.

Subject headings: Experimental tests of gravitational theories; Modified theories of gravity; Celestial mechanics; Orbit determination and improvement; Ephemerides, almanacs, and calendars

1. Introduction

General Relativity (GR) has passed with excellent results many observational tests: a satisfactory agreement comes both from Solar System tests and from binary pulsars observations. As a matter of fact (see e.g. (Will 2006)), the current values of the PPN parameters are in agreement with GR predictions and, consequently, Einstein’s theory is the classical theory of gravitational interactions accepted nowadays.

However, observations seem to question the general relativistic model of gravitational interactions on large scales. On the one hand, the data coming from the rotation curves of spiral galaxies (Binney and Tremaine 1987) cannot be explained on the basis of Newtonian gravity or GR: the existence of a peculiar form of matter is postulated to reconcile the theoretical model with observations, i.e. dark matter, which is supposed to be a cold and pressureless medium, whose distribution is that of a spherical halo around the galaxies. Furthermore, dark matter can explain the mass discrepancy in galactic clusters (Clowe et al. 2006). On the other hand, a lot of observations, such as the light curves of the type Ia supernovæ and the cosmic microwave background (CMB) experiments (Riess et al. 1998; Perlmutter et al. 1999; Bennet et al. 2003), firmly state that our Universe is now undergoing a phase of accelerated expansion. Actually, the present acceleration of the Universe cannot be explained, within GR, unless the existence of a cosmic fluid having exotic properties is postulate, i.e. dark energy or introducing a cosmological constant which, in turn, brings about other problems, concerning its nature and origin (Peebles and Ratra 2003).

The main problem one has to face with dark matter and dark energy (or the cosmological constant) is understanding their nature, since they are introduced as *ad hoc* gravity sources in GR or its weak-field limit, Newtonian gravity.

In order to explain the observations another possibility exists: the query for dark matter and dark energy points out the failure of GR (and its approximation, Newtonian

gravity) to deal with gravitational interaction at galactic, intergalactic and cosmological scales. The latter viewpoint led to the introduction of various modified gravity models.

In this paper, we are concerned with the so called $f(R)$ theories of gravity, where the gravitational Lagrangian depends on a function f of the scalar curvature R (see (Capozziello and Francaviglia 2007; Sotiriou and Faraoni 2008) and references therein). These theories are also referred to as “extended theories of gravity”, since they naturally generalize GR: in fact, when $f(R) = R$ the action reduces to the usual Einstein-Hilbert action, and Einstein’s theory is obtained. These theories can be studied in the metric formalism, where the action is varied with respect to metric tensor, and in the Palatini formalism, where the action is varied with respect to the metric and the affine connection, which are supposed to be independent from one another (actually, there is also the possibility that the matter part of the action depends on the affine connection, and is then varied with respect to it: this is the so-called metric-affine formalism, but we are not concerned with this approach in this paper). In general, the two approaches are not equivalent: the solutions of the Palatini field equations are a subset on solutions of the metric field equations (Magnano 1994).

Actually, $f(R)$ theories provide cosmologically viable models, where both the inflation phase and the accelerated expansion are reproduced (see (Nojiri and Odintsov 2007, 2008a,b) and references therein). Furthermore, they have been used to explain the rotation curves of galaxies without need for dark matter (Capozziello et al. 2007a; Frigerio Martins and Salucci 2007).

However, because of the excellent agreement of GR with Solar System and binary pulsar observations, every theory that aims at explaining galaxies dynamics and the accelerated expansion of the Universe, should reproduce GR at the Solar System scale, i.e. in a suitable weak-field limit. In other words, also for $f(R)$ theories the constraint

holds to have correct Newtonian and post-Newtonian limits. This issue has been lively debated in the recent literature, where different approaches to the problem have been taken into account, both in the Palatini and metric formalism. A thorough discussion can be found in the recent review by Sotiriou and Faraoni (2008). In summary, with respect to the weak-field tests and, more in general, the non cosmological solutions (see e.g. (Faraoni 2008)), it seems that there are difficulties in considering Palatini $f(R)$ gravity as a viable theory because the Cauchy problem is ill-posed and, furthermore, curvature singularities arise when dealing with simple stellar models; as for metric $f(R)$ gravity, there are models that are in agreement with the weak-field tests, but it seems that curvature singularities exist, in this case, for compact relativistic stars.

Without going into the details of this interesting debate, in this paper we want to test some models of $f(R)$ gravity that can be used to describe the gravitational field of the Sun. In particular, we are going to examine the impact that the modifications of the gravitational field of GR have on the Solar System dynamics. We apply a perturbative approach to compare the $f(R)$ -induced secular precession of the longitude of the pericentre ϖ of a test particle with the latest determinations of the corrections to the usual perihelion precessions coming from fits of huge planetary data sets with various versions of the EPM ephemerides (Pitjeva 2005a,b, 2006, 2008a,b).

The paper is organized as follows: in Section 2 we briefly review the theoretical formalism of $f(R)$ gravity, both in the metric and Palatini approach, then, in Section 3 we outline a general approach to the perturbations of the gravitational field of GR, due to the non-linearity of the gravity Lagrangian. In Section 4 we compare the theoretical predictions with the observations. Finally, discussion and conclusions are in Section 5.

2. The field equations of $f(R)$ gravity

In this Section, we introduce the field equations of $f(R)$ gravity. We shall consider both the metric and the Palatini approach (see, e.g., (Capozziello and Francaviglia 2007) and (Sotiriou and Faraoni 2008)).

The equations of motion of $f(R)$ extended theories of gravity can be obtained by a variational principle, starting from the action:

$$A = A_{\text{grav}} + A_{\text{mat}} = \int [\sqrt{g}f(R) + 2\chi L_{\text{mat}}(\psi, \nabla\psi)] d^4x. \quad (1)$$

The gravitational part of the Lagrangian is represented by a function $f(R)$ of the scalar curvature R . The total Lagrangian contains also a first order matter part L_{mat} , functionally depending on matter fields Ψ , together with their first derivatives, equipped with a gravitational coupling constant $\chi = \frac{8\pi G}{c^4}$. In the metric formalism, Γ is supposed to be the Levi-Civita connection of g and, consequently, the scalar curvature R has to be intended as $R \equiv R(g) = g^{\alpha\beta}R_{\alpha\beta}(g)$. On the contrary, in the Palatini formalism the metric g and the affine connection Γ are supposed to be independent, so that the scalar curvature R has to be intended as $R \equiv R(g, \Gamma) = g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$, where $R_{\mu\nu}(\Gamma)$ is the Ricci-like tensor of the connection Γ .

In the metric formalism the action (1) is varied with respect to the metric g , and one obtains the following field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\square)f'(R) = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (2)$$

where $f'(R) = df(R)/dR$, and $T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$ is the standard minimally coupled matter energy-momentum tensor. The contraction of the field equations (2) with the metric tensor leads to the scalar equation

$$3\square f'(R) + f'(R)R - 2f(R) = \frac{8\pi G}{c^4}T, \quad (3)$$

where T is the trace of the energy-momentum tensor. Eq. (3) is a differential equation for the scalar curvature R .

In the Palatini formalism, by independent variations with respect to the metric g and the connection Γ , we obtain the following equations of motion:

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (4)$$

$$\nabla^\Gamma_\alpha[\sqrt{g}f'(R)g^{\mu\nu}] = 0, \quad (5)$$

where ∇^Γ means covariant derivative with respect to the connection Γ . Actually, it is possible to show (Ferraris et al. 1993, 1994) that the manifold M , which is the model of the space-time, can be a posteriori endowed with a bi-metric structure (M, g, h) equivalent to the original metric-affine structure (M, g, Γ) , where Γ is assumed to be the Levi-Civita connection of h . The two metrics are conformally related by

$$h_{\mu\nu} = f'(R)g_{\mu\nu}. \quad (6)$$

The equation of motion (4) can be supplemented by the scalar-valued equation obtained by taking the contraction of (4) with the metric tensor:

$$f'(R)R - 2f(R) = \frac{8\pi G}{c^4}T. \quad (7)$$

Equation (7) is an algebraic equation for the scalar curvature R .

In order to compare the predictions of $f(R)$ gravity with Solar System data, we have to consider the solutions of the field equations (2),(4),(5) - supplemented by the constraints (3),(7) - in vacuum, since tests are based on the observations of the dynamics of the planets in the gravitational field of the Sun.

In particular, the vacuum field equations in the metric approach read

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\square)f'(R) = 0, \quad (8)$$

supplemented with the scalar equation

$$3\square f'(R) + f'(R)R - 2f(R) = 0. \quad (9)$$

In the Palatini approach, the field equations become

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = 0, \quad (10)$$

$$\nabla^\Gamma_\alpha[\sqrt{g}f'(R)g^{\mu\nu}] = 0, \quad (11)$$

and they are supplemented by the scalar equation

$$f'(R)R - 2f(R) = 0 \quad (12)$$

We want to point out some general features of the scalar equations (9) and (12), which can help to understand the differences between the vacuum solutions in the two formalisms.

In Palatini $f(R)$ gravity, the trace equation (12) is an algebraic equation for R , which admits constant solutions $R = c_i$ (Ferraris et al. 1993), and it is identically satisfied if $f(R)$ is proportional to R^2 . As a consequence, it is easy to verify that (if $f'(R) \neq 0$) the field equations become

$$R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu} \quad (13)$$

which are the same as GR field equations with a cosmological constant. In other words, in the Palatini formalism, in vacuum, we can have only solutions that describe space-times with constant scalar curvature R . Summarizing, eq. (13) suggests that all GR solutions with cosmological constant are solutions of vacuum Palatini field equations: the function $f(R)$ determines the solutions of algebraic equation (12).

In metric $f(R)$ gravity the trace equation (9) is a differential equation for R : this means that, in general, it admits more solutions than the corresponding Palatini equation. In particular, we notice that if $R = \text{constant}$ we obtain the Palatini case: so for a given $f(R)$ function, in vacuum, the solutions of the field equations of Palatini $f(R)$ gravity are

a subset of the solutions of the field equations of metric $f(R)$ gravity (Magnano 1994); however, in metric $f(R)$ gravity, vacuum solutions with variable R are allowed too (see, e.g., (Multamaki and Vilja 2006)).

3. Corrections to the gravitational potential

We have seen in the previous Section that, when $f(R) \neq R$, the field equations of $f(R)$ gravity are different from those of GR. Thus, it is evident that the solutions of such modified field equations describing the gravitational field of a point-like mass (e.g. the Sun) contain corrections to the GR solutions, both at Newtonian and post-Newtonian level. However, these corrections have to be small enough not to contradict the known tests of GR. Thus, it is possible to treat them perturbatively to evaluate their impact on the dynamics of the Solar System planets.

In this Section we want to outline the general procedure that we are going to apply to some solutions of $f(R)$ gravity that can be used to describe the gravitational field of the Sun, in order to compare the predictions of these gravity models with the existing data.

In general, we are going to deal with spherically symmetrical metrics, describing the space-time around a point-like mass M , which can be endowed with proper angular momentum \mathbf{J} . The weak-field and slow-motion approximations of these metrics will be sufficient for our purposes. Generally speaking, this means that the deviations from GR will be linear in some parameters deriving from the specific $f(R)$ gravity model.

On using spherical isotropic coordinates, these metric have the general form¹

$$ds^2 = A(r)dt^2 + B(r)(dr^2 + r^2d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2) + 2C(r)\sin^2\theta dt d\varphi, \quad (14)$$

¹If not otherwise stated, here and henceforth we use units such that $G = c = 1$.

where the angular momentum \mathbf{J} is assumed to be perpendicular to the $\theta = \pi/2$ plane.

The gravitational (scalar) potential $\Phi(r)$ is read from the $A(r)$ function

$$A(r) = 1 + 2\Phi(r). \quad (15)$$

According to what stated before, we expect a gravitational potential in the form

$$\Phi(r) = \Phi^N(r) + \Delta\Phi(r), \quad (16)$$

where $\Phi^N(r) = -\frac{M}{r}$ is the Newtonian potential of a point-like mass M , and $\Delta\Phi(r) \ll \Phi_N(r)$ is a correction vanishing for $f(R) \rightarrow R$.

The $C(r)$ function accounts for the presence of the so-called gravito-magnetic effects (Ruggiero and Tartaglia 2002; Mashhoon 2007) induced by the rotation of the source of the gravitational field. In GR $C(r)$ is given by the suitable component of the gravito-magnetic vector potential of a gravito-magnetic dipole, i.e. $A_\varphi^{\text{GR}}(r) = -\frac{2J}{r}$ (see e.g. (Mashhoon 2007)). As a consequence, we expect that the $C(r)$ function has the form

$$C(r) = A_\varphi^{\text{GR}}(r) + \Delta A_\varphi(r) \quad (17)$$

where, again, $\Delta A_\varphi(r) \ll A_\varphi(r)^{\text{GR}}$ is a correction vanishing for $f(R) \rightarrow R$.

We can use the gravito-electromagnetic (Ruggiero and Tartaglia 2002; Mashhoon 2007) formalism to describe the total perturbing acceleration felt by a test particle in the metric (14)

$$\mathbf{W} = -\mathcal{E}^G - 2\mathbf{v} \times \mathcal{B}^G \quad (18)$$

where

$$\mathcal{E}^G = -\frac{d\Delta\Phi(r)}{dr}\hat{\mathbf{r}} \quad (19)$$

and

$$\mathcal{B}^G = \nabla \times \mathbf{A}, \quad \mathbf{A} = \frac{\Delta A_\varphi}{r \sin \theta} \quad (20)$$

Hence, given the perturbing acceleration (18), we can calculate its effects on planetary motions within standard perturbative schemes (see, e.g., Roy (2005)). We may use the Gauss equations for the variations of the elements, which enable us to study the perturbations of the Keplerian orbital elements due to a generic perturbing acceleration, whatever its physical origin is. The Gauss equations for the variations of the semi-major axis a , the eccentricity e , the inclination i , the longitude of the ascending node Ω , the argument of pericentre ω and the mean anomaly \mathcal{M} of a test particle in the gravitational field of a body M are (Roy 2005)

$$\frac{da}{dt} = \frac{2}{\bar{n}\sqrt{1-e^2}} \left[eW_r \sin v + W_\tau \left(\frac{p}{r} \right) \right], \quad (21)$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{\bar{n}a} \left\{ W_r \sin v + W_\tau \left[\cos v + \frac{1}{e} \left(1 - \frac{r}{a} \right) \right] \right\}, \quad (22)$$

$$\frac{di}{dt} = \frac{1}{\bar{n}a\sqrt{1-e^2}} W_\nu \left(\frac{r}{a} \right) \cos(\omega + v), \quad (23)$$

$$\frac{d\Omega}{dt} = \frac{1}{\bar{n}a \sin i \sqrt{1-e^2}} W_\nu \left(\frac{r}{a} \right) \sin(\omega + v), \quad (24)$$

$$\frac{d\omega}{dt} = -\cos i \frac{d\Omega}{dt} + \frac{\sqrt{1-e^2}}{\bar{n}ae} \left[-W_r \cos v + W_\tau \left(1 + \frac{r}{p} \right) \sin v \right], \quad (25)$$

$$\frac{d\mathcal{M}}{dt} = \bar{n} - \frac{2}{\bar{n}a} W_r \left(\frac{r}{a} \right) - \sqrt{1-e^2} \left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt} \right), \quad (26)$$

in which $\bar{n} = 2\pi/P$ is the mean motion², P is the test particle's orbital period, v is the true anomaly counted from the pericentre, $p = a(1 - e^2)$ is the semilatus rectum of the Keplerian ellipse, W_r , W_τ , W_ν are the radial, transverse (in-plane components) and normal (out-of-plane component) projections of the perturbing acceleration \mathbf{W} , respectively, on the orthonormal frame $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\nu}}\}$ comoving with the particle.

For our purposes it is useful to consider the longitude of the pericenter $\varpi = \omega + \cos i \Omega$.

²For an unperturbed Keplerian ellipse it is $\bar{n} = \sqrt{GM/a^3}$.

The Gauss equation for its variation under the action of an entirely radial perturbing acceleration W_r is

$$\frac{d\varpi}{dt} = -\frac{\sqrt{1-e^2}}{\bar{n}ae} W_r \cos v. \quad (27)$$

After being evaluated onto the unperturbed Keplerian ellipse, the acceleration (18) must be inserted into eq. (27); then, the average over one orbital period P must be performed. To this end it is useful also to recall the following relations where also the eccentric anomaly E is used

$$\left\{ \begin{array}{l} r = a(1 - e \cos E), \\ dt = \frac{(1-e \cos E)}{\bar{n}} dE, \\ \cos v = \frac{\cos E - e}{1 - e \cos E}, \\ \sin v = \frac{\sin E \sqrt{1-e^2}}{1 - e \cos E}. \end{array} \right. \quad (28)$$

In fact, what we aim at is evaluating the perturbations induced on the longitudes of the perihelia by the corrections to the gravitational field due to $f(R)$ gravity, in order to compare them with the latest observational determinations. The astronomer E.V. Pitjeva (Institute of Applied Astronomy, Russian Academy of Sciences, St. Petersburg) processed almost one century of data of different types for the major bodies of the Solar System to improve the EPM planetary ephemerides (Pitjeva 2005a, 2008a,b). Among other things, she simultaneously estimated corrections to the secular rates of the longitudes of perihelia ϖ of the inner (Pitjeva 2005b) and of some of the outer (Pitjeva 2008a,b) planets of the Solar System as fit-for parameters of global solutions in which she contrasted, in a least-square way, the observations to their predicted values computed with a complete set of dynamical force models including all the known Newtonian (solar quadrupole mass moment J_2 , N -body interactions with the major planets, 301 biggest asteroids, massive ring of the

small asteroids, 20 largest trans-Neptunian objects and massive ring for the other ones) and Einsteinian³ features of motion. As a consequence, any force that is not present in Newtonian gravity or GR is, in principle, accounted for by the estimated corrections to the usual apsidal precessions. For the sake of completeness, we reproduce in tables 1 and 2 the estimated perihelia extra-precessions for inner and outer planets, respectively.

What we want to do is to see whether the estimated perihelia extra-precessions are compatible with the perturbations of the gravitational field deriving from the non-linearity of $f(R)$.

Now, let us briefly outline how we are going to put $f(R)$ gravity on the test. In general a correction to the gravitational field of GR due to the non linearity of the gravity Lagrangian, in the weak-field and slow motion approximation, can be parameterized in terms of a parameter κ , where $\kappa \rightarrow 0$ as far as $f(R) \rightarrow R$. In other words, κ is a measure of the non-linearity of the Lagrangian. Let $\mathcal{P}(f(R))$ be the prediction of a certain effect induced by these modified gravity models, e.g. the secular precession of the perihelion of a planet: for all the $f(R)$ models that we are going to consider below, it turns out that

$$\mathcal{P}(f(R)) = \kappa g(a, e), \quad (29)$$

where g is a function of the system's orbital parameters a (semi-major axis) and e (eccentricity); such g is a peculiar consequence of the $f(R)$ gravity model. Now, let us take the ratio of $\mathcal{P}(f(R))$ for two different systems A and B, e.g. two Solar System's planets: $\mathcal{P}_A(f(R))/\mathcal{P}_B(f(R)) = g_A/g_B$. The model's parameter κ has now been canceled, but we still have a prediction that retains a peculiar signature of that model, i.e. g_A/g_B . Of course, such a prediction is valid if we assume κ is not zero, which is just the case both theoretically (only if $f(R) = R$ then $\kappa = 0$) and observationally because κ is usually determined by other

³The general relativistic gravito-magnetic Lense-Thirring force has not yet been modeled.

Table 1: Inner planets. First row: estimated perihelion extra-precessions, from Table 3 of (Pitjeva 2005b). The quoted errors are not the formal ones but are realistic. The units are arc-seconds per century (" cy⁻¹). Second row: semi-major axes, in Astronomical Units (AU). Their formal errors are in Table IV of (Pitjeva 2005a), in m. Third row: eccentricities. Fourth row: orbital periods in years.

	Mercury	Earth	Mars
$\langle \dot{\varpi} \rangle$ (" cy ⁻¹)	-0.0036 ± 0.0050	-0.0002 ± 0.0004	0.0001 ± 0.0005
a (AU)	0.387	1.000	1.523
e	0.2056	0.0167	0.0934
P (yr)	0.24	1.00	1.88

Table 2: Outer planets. First row: estimated perihelion extra-precessions (Pitjeva 2006). The quoted uncertainties are the formal, statistical errors re-scaled by a factor 10 in order to get the realistic ones. The units are arc-seconds per century (" cy⁻¹). Second row: semi-major axes, in Astronomical Units (AU). Their formal errors are in Table IV of (Pitjeva 2005a), in m. Third row: eccentricities. Fourth row: orbital periods in years.

	Jupiter	Saturn	Uranus
$\langle \dot{\varpi} \rangle$ (" cy ⁻¹)	0.0062 ± 0.036	-0.92 ± 2.9	0.57 ± 13.0
a (AU)	5.203	9.537	19.191
e	0.0483	0.0541	0.0471
P (yr)	11.86	29.45	84.07

independent long-range astrophysical/cosmological observations. Otherwise, one would have the meaningless prediction 0/0. The case $\kappa = 0$ (or $\kappa \leq \bar{\kappa}$, i.e. when κ is negligibly small) can be, instead, usually tested by taking one perihelion precession at a time. If we have observational determinations \mathcal{O} for A and B of the effect considered above such that they are affected also⁴ by the $f(R)$ gravity model (it is just the case for the purely phenomenologically estimated corrections to the standard Newton-Einstein perihelion precessions, since any $f(R)$ gravity model has not been included in the dynamical force models of the ephemerides adjusted to the planetary data in the least-square parameters' estimation process by Pitjeva (Pitjeva 2005a,b)), we can construct $\mathcal{O}_A/\mathcal{O}_B$ and compare it with the prediction for it by $f(R)$, i.e. with g_A/g_B . Note that $\delta\mathcal{O}/\mathcal{O} > 1$ only means that \mathcal{O} is compatible with zero, being possible a nonzero value smaller than $\delta\mathcal{O}$. Thus, it is perfectly meaningful to construct $\mathcal{O}_A/\mathcal{O}_B$. Its uncertainty will be conservatively evaluated as $|1/\mathcal{O}_B|\delta\mathcal{O}_A + |\mathcal{O}_A/\mathcal{O}_B^2|\delta\mathcal{O}_B$. As a result, $\mathcal{O}_A/\mathcal{O}_B$ will be compatible with zero. Now, the question is: Is it the same for g_A/g_B as well? If yes, i.e. if

$$\frac{\mathcal{O}_A}{\mathcal{O}_B} = \frac{\mathcal{P}_A(f(R))}{\mathcal{P}_B(f(R))} \quad (30)$$

within the errors, or, equivalently, if

$$\left| \frac{\mathcal{O}_A}{\mathcal{O}_B} - \frac{\mathcal{P}_A(f(R))}{\mathcal{P}_B(f(R))} \right| = 0 \quad (31)$$

within the errors, the $f(R)$ gravity model examined can still be considered compatible with the data, otherwise it is seriously challenged.

In next Section some solutions of $f(R)$ gravity that can be used to describe the gravitational field of the Sun will be tested according to the procedure that we have just

⁴If they are differential quantities constructed by contrasting observations to predictions obtained by analytical force models of canonical Newtonian/Einsteinian effects, \mathcal{O} are, in principle, affected also by the mis-modeling in them.

described.

4. $f(R)$ weak-field solutions

In this Section we introduce some solutions of $f(R)$ gravity that have been used in the literature to describe the weak gravitational field, and that can be considered as suitable models of the gravitational field of the Sun. We consider these modified gravitational fields and, within the perturbative scheme outlined above, compare the theoretical predictions with the estimated extra-precessions of the planetary perihelia.

4.1. Power law corrections

Starting from a Lagrangian of the form $f(R) = f_0 R^n$, Capozziello et al. (2007a), in the metric approach, look for solutions describing the gravitational field of a point-like source, in order to reproduce the galaxies rotation curves without need for dark matter. As a result, they obtain the following power-law form for the gravitational potential:

$$\Phi(r) = -\frac{M}{r} \left[1 + \left(\frac{r}{r_c} \right)^\beta \right]. \quad (32)$$

The deviation from the Newtonian potential is parameterized by a power law, with two free parameters β and r_c . In particular, β is related to n , i.e. the exponential of the Ricci scalar in $f(R) = f_0 R^n$.

In this case, the correction to the gravitational potential is

$$\Delta\Phi(r) = -\frac{M}{r} \left(\frac{r}{r_c} \right)^\beta, \quad (33)$$

which clearly leads to the radial acceleration

$$W_r = \frac{(\beta - 1)M}{r_c^\beta} r^{\beta-2} \quad (34)$$

It yields the following perihelion precession (Iorio and Ruggiero 2008a)

$$\langle \dot{\varpi} \rangle = \frac{(\beta - 1)\sqrt{M}}{2\pi r_c^\beta} a^{\beta-\frac{3}{2}} G(e; \beta), \quad (35)$$

with $G(e_A; \beta)/G(e_B; \beta) \approx 1$ for all the planets of the Solar System.

Capozziello et al. (2007a) find $\beta = 0.817$ from a successful fit of several galactic rotation curves with no dark matter; it is ruled out by comparing for several pairs of planets $\Delta\dot{\varpi}_A/\Delta\dot{\varpi}_B$ to $\mathcal{P}_A/\mathcal{P}_B$ obtained from eq. (35) (Iorio and Ruggiero 2008a).

4.2. Schwarzschild-de Sitter-like corrections

The field equations (4-5) and the structural equation (7), in the Palatini formalism have the spherically symmetrical solution (see (Allemandi et al. 2005) and (Ruggiero and Iorio 2007)):

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{k}{3}r^2\right) dt^2 - \left(1 + \frac{2M}{r} - \frac{k}{6}r^2\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (36)$$

$k = c_i/4$, where $R = c_i$ is any of the solutions of the structural equation (7). In this case, we may write

$$\Phi(r) = -\frac{M}{r} + \frac{kr^2}{6}. \quad (37)$$

In this case, the perturbing potential is

$$\Delta\Phi(r) = \frac{kr^2}{6}, \quad (38)$$

and the induced the perturbing acceleration is

$$W_r = -\frac{1}{3}kr. \quad (39)$$

As any other Hooke-type extra-acceleration, eq. (39) induces a secular perihelion precession (Kerr et al. 2003; Iorio 2008a)

$$\langle \dot{\varpi} \rangle \propto \frac{k}{n} = k \sqrt{\frac{a^3}{M}}. \quad (40)$$

By using eq. (40) to construct $\mathcal{P}_A/\mathcal{P}_B$ for different pairs of Solar System's planets and comparing them to $\Delta\dot{\varpi}_A/\Delta\dot{\varpi}_B$ yield a negative answer (Iorio 2008a).

4.3. Logarithmic corrections

Sobouti (2007) aims at determining a $f(R)$ able to explain the rotation curves of the galaxies obtained. In particular, working in the metric approach, solutions with R variable with the radial coordinate r are obtained. In this context, the gravitational potential reads:

$$\Phi(R) = -\frac{M}{r} + \frac{\alpha}{2} + \frac{\alpha}{2} \ln(r/2M). \quad (41)$$

The parameter α can be related to Modified Newtonian Dynamics (MOND, see e.g. (Milgrom 1983)) characteristic acceleration A_0 .

The Logarithmic correction to the Newtonian gravitational potential assumes the form

$$\Delta\Phi(r) = -\gamma M \ln\left(\frac{r}{r_0}\right), \quad (42)$$

and leads to a perturbing radial acceleration

$$W_r = \frac{\gamma M}{r}. \quad (43)$$

In particular, in order to agree with the potential (41), we must set $\gamma = -\alpha/2$, $r_0 = 2M$.

This kind of acceleration has been treated by Iorio and Ruggiero (2008a) with the approach outlined here getting negative answers.

4.4. Yukawa-like corrections

In different works, both in the Palatini (Hamity and Barraco 1993; Barraco et al. 1996) and metric approach (e.g. see (Pechlaner and Sexl 1966; Stelle 1978; Capozziello et al. 2007b)) Yukawa-like corrections are obtained. They lead to a gravitational potential in the form

$$\Phi(r) = -\frac{M}{r} \left[1 + \alpha \exp\left(-\frac{r}{\lambda}\right) \right] \quad (44)$$

The parameter α is related to the strength of the correction, while λ is related to the range of the modified potential.

The Yukawa correction to the Newtonian potential

$$\Delta\Phi(r) = -\frac{M\alpha}{r} \exp\left(-\frac{r}{\lambda}\right) \quad (45)$$

yields an entirely radial extra-acceleration

$$W_r = -\frac{M\alpha}{r^2} \left(1 + \frac{r}{\lambda} \right) \exp\left(-\frac{r}{\lambda}\right) \quad (46)$$

By only assuming $\lambda \gg ae$, i.e. Yukawa-type long-range modifications of gravity, it is possible to obtain useful approximated expressions for the induced perihelion precession which, in turn, allow to obtain (Iorio 2007a)

$$\lambda = \frac{a_B - a_A}{\ln\left(\sqrt{\frac{a_B}{a_A}} \frac{\Delta\dot{\varpi}_A}{\Delta\dot{\varpi}_B}\right)} \quad (47)$$

for the range and

$$\alpha = \frac{2\lambda^2 \Delta\dot{\varpi}}{\sqrt{Ma}} \exp\left(\frac{a}{\lambda}\right). \quad (48)$$

for the strength. By using A = Earth, B = Mercury in eq. (47) one gets $\lambda = 0.182 \pm 0.183$ AU; such a value for λ , the data of Venus and eq. (48) yield $\alpha = (-1 \pm 4) \times 10^{-11}$ (Iorio 2008b).

4.5. Gravito-magnetic effects

We have shown that the vacuum solutions of General Relativity with a cosmological constant can be used in Palatini $f(R)$ gravity. In particular, the Kerr-de Sitter solution, which describes a rotating black-hole in a space-time with a cosmological constant (Demianski 1973; Carter 1973; Kerr et al. 2003; Kraniotis 2004, 2005, 2007), can be used to investigate Gravito-magnetic effects in extended theories of gravity.

In particular, the weak-field and slow-motion approximation of the Kerr-de Sitter is (Iorio and Ruggiero 2008b)

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{k}{3}r^2\right) dt^2 - \left(1 + \frac{2M}{r} - \frac{k}{6}r^2\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + \\ + 2\frac{J}{M} \left(\frac{2M}{r} + \frac{k}{3}r^2 + \frac{5}{6}Mkr\right) \sin^2 \theta d\phi dt. \quad (49)$$

We obtain the following expression for the perturbing gravito-magnetic potential

$$\Delta A_\varphi = \frac{J}{M} \left(\frac{k}{3}r^2 + \frac{5}{6}Mkr\right) \sin^2 \theta. \quad (50)$$

Furthermore, the perturbing acceleration is

$$\mathbf{W} = -2\mathbf{v} \times \mathbf{B}^G, \quad (51)$$

where the gravito-magnetic field \mathbf{B}^G is

$$\mathbf{B}^G = \frac{Jk}{3M} \hat{\mathbf{J}} + \frac{5Jk}{12} \frac{\left[\hat{\mathbf{J}} + (\hat{\mathbf{J}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}\right]}{r}. \quad (52)$$

The resulting orbital effects are (Iorio and Ruggiero 2008b)

$$\langle \dot{a} \rangle = 0, \quad (53)$$

$$\langle \dot{e} \rangle = 0, \quad (54)$$

$$\langle \dot{i} \rangle = 0, \quad (55)$$

$$\langle \dot{\Omega} \rangle = \frac{Jk}{3M} \left(1 + \frac{5M}{2a} \right), \quad (56)$$

$$\langle \dot{\omega} \rangle = -\frac{2Jk \cos i}{3M} \left(1 + \frac{5M}{4a} \right), \quad (57)$$

$$\langle \dot{\mathcal{M}} \rangle = \bar{n} + \frac{5Jk \cos i}{3M} \left(1 + \frac{M}{a} \right). \quad (58)$$

In the calculation we have neglected terms of order $\mathcal{O}(e^2)$.

By using the corrections $\Delta\dot{\varpi}$ separately for each Solar System's planet one gets $k \leq 10^{-29} \text{ m}^{-2}$.

For all the Solar System's planets the perihelion rate can be satisfactorily approximated by

$$\langle \dot{\omega} \rangle \approx -\frac{2Jk \cos i}{3M}. \quad (59)$$

Since $\cos i_A / \cos i_B \approx 1$ for every pair of planets A and B, in this case $\mathcal{P}_A / \mathcal{P}_B \approx 1$; this possibility is ruled out by $\mathcal{O}_A / \mathcal{O}_B = \Delta\dot{\varpi}_A / \Delta\dot{\varpi}_B$, as in the case of the DGP (Dvali et al. 2000) braneworld scenario (Iorio 2007b).

5. Discussion and Conclusions

In this paper we have considered some solutions of $f(R)$ gravity, both in the Palatini and metric formalism, that can be used to describe the weak gravitational field around the Sun. In particular, we have focused on the impact that the modifications of the GR gravitational field, due to the non-linearity of $f(R)$, have on the Solar System dynamics.

We have considered that these modifications have to be small in order not to contradict the known tests of GR and, as a consequence, we have treated them as perturbations. Thus, we have applied a perturbative approach to compare the $f(R)$ -induced secular effects with the latest observationally determinations coming from various versions of the EPM planetary ephemerides. In particular, we have considered the ratios of the corrections to the standard secular precessions of the longitudes of perihelia estimated by E.V. Pitjeva for several pairs of planets in the Solar System. For all the models that we have considered (power law, Hooke-like force, logarithmic corrections, Yukawa-like force, gravito-magnetic effects) our results show that the perturbations deriving from the non-linearity of $f(R)$ are not compatible with the currently available apsidal extra-precessions of the Solar System planets. Moreover, the hypothesis that the examined $f(R)$ -induced perturbations are zero, which cannot be tested by definition with our approach, is compatible with each perihelion extra-rate separately.

This might suggest that, on the one hand, the $f(R)$ -induced secular effects cannot explain the observed extra-precessions and that, on the other hand, the effects of the non-linearity of the gravity Lagrangian are important on length scales much larger than the Solar System (e.g. on the cosmological scale) and their effects on local physics are probably negligible. It will be important to repeat such tests if and when other teams of astronomers will independently estimate their own corrections to the standard secular precessions of the perihelia.

Acknowledgments

The authors would like to thank Prof. P. M. Lavrov for his kind invitation to contribute to the volume *The Problems of Modern Cosmology*, on the occasion of the 50th birthday of Professor Sergei D. Odintsov. L.I. gratefully thanks E.V. Pitjeva (Institute of

Applied Astronomy, Russian Academy of Sciences, St. Petersburg) for useful and important information concerning her unpublished determinations of the perihelion precessions.

M.L.R. acknowledges financial support from the Italian Ministry of University and Research (MIUR) under the national program “Cofin 2005” - *La pulsar doppia e oltre: verso una nuova era della ricerca sulle pulsar.*

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